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The Pluriharmonic Toeplitz Operators on the Polydisk[1]

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In this paper we completely characterize the compact semi-commutator of two Toeplitz operators with bounded pluriharmonic symbols on the Bergman space of the polydisk. Several necessary and sufficient conditions are obtained for the commutator of two Toeplitz operators with bounded pluriharmonic symbols on the polydisk to be compact. © 2001 Academic Press

Key Words: Toeplitz operator; Bergman space; polydisk.

1. PRELIMINARIES

Let D be the open unit disk in C. Its boundary is the circle T. The polydisk D^n and the torus T^n are the subsets of C^n which are Cartesian products of n copies D and T, respectively. Let $dA(z)$ be the normalized volume measure on D^n. There is an orthogonal projection P from $L^2(D^n, dA)$ onto $L_a^2(D^n)$. The Toeplitz operator with symbol f in $L^\infty(D^n)$ is defined by $T_f h = P(fh)$, for all $h \in L_a^2(D^n)$ and the Hankel operator with symbol f is defined by $H_f h = (I - P)(fh)$, for all $h \in L_a^2(D^n)$.

Axler and Čučković [1] completely characterize commuting Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk. The function theory on the polydisk D^n is quite different from the function theory on the unit disk [4]. One may expect that there should exist some differences in operator theory on the Bergman spaces between the polydisk and the disk. Sun and Zheng showed that two analytic Toeplitz operators essentially doubly commute if and only if they doubly commute on the Bergman space of the polydisk [7]. But this is false on the disk [2, 8]. In this paper we will show that a semi-commutator of plurihar-

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monic Toeplitz operators is compact if and only if it is zero. We also will obtain partial results for the compactness of $T_f T_g - T_g T_f$.

Observe that T^n is only a small part of the boundary of D^n if $n > 1$. But it is an important part and it is called the distinguished boundary of D^n. T^n is also a compact group (with componentwise multiplication as group operation) and as such carries a Haar measure. Its dual group is Z^n where Z is the integer group. As in [6] we consider that multiple Fourier series on the n-norms T^n can be viewed as the Fourier transformation on $L^1(T^n)$. For f in $L^1(T^n)$ the Fourier transformation is given by

$$\tilde{f}(\mathbf{m}) = \int_{T^n} f(x_1, \dots, x_n) e^{i(\mathbf{m}, \mathbf{x})}\, d\sigma(x_1) \cdots d\sigma(x_n),$$

where $\mathbf{m} = (m_1, \dots, m_n) \in Z^n$ and $(\mathbf{m}, \mathbf{x}) = \sum_{i=1}^n m_i x_i$ and $\sigma_i(x_i)$ is the normalized Haar measure on T for $i = 1, \dots, n$. By Theorem 1.7 in [6], the Fourier transformation is injective; i.e., if $f \in L^1(T^n)$ and $\tilde{f}(m) = 0$ for all $m \in Z^n$, then $f(x) = 0$ for almost all $x \in T^n$. Since for any $z \in D^n$, the pointwise evaluation of functions in $L_a^2(D^n)$ at z is a bounded functional, there is a function K_z in $L_a^2(D^n)$ such that $f(z) = \langle f, K_z \rangle$ for all f in $L_a^2(D^n)$. K_z is called the Bergman reproducing kernel. Let K_{z_1} denote the Bergman producing kernel $1/(1 - \bar{z}_1 w_1)^2$ of the Bergman space $L_a^2(D)$ of the unit disk at the point z_1 in D, and k_{z_1} the normalized Bergman reproducing kernel of the Bergman space $L_a^2(D)$ at the point $z_1 \in D$. We use z to denote the vector $(z_1, \dots, z_n)$ in C^n of an n-dimensional complex plane. It is easy to check that the reproducing kernel K_z of the Bergman space $L_a^2(D^n)$ of the polydisk is the multiple product $\prod_{i=1}^n K_{z_i}(w_i)$ of the Bergman kernel of the unit disk D. So the normalized producing kernel k_z of $L_a^2(D^n)$ is also the multiple product $\prod_{i=1}^n k_{z_i}$ of the normalized kernel of the unit disk. It is well known that k_z weakly converges to zero in $L_a^2(D^n)$ as z tends to the boundary of D^n.

In addition, the producing kernel K_z has the nice property

$$T_{\bar{f}} K_z = \overline{f(z)} K_z \tag{1.1}$$

for z in D^n if f is in $L_a^2(D^n)$.

Let Z_+^n denote the subset $\{\mathbf{m} \in Z^n \colon m_i \geq 0\ \forall i = 1, \dots, n\}$ of Z^n. For a function f in the Hardy space $H^2(D^n)$, we write the power series of f as

$$f(z) = \sum_{\mathbf{m} \in Z_+^n} a_{\mathbf{m}} z^{\mathbf{m}},$$

where $a_{\mathbf{m}}$ is a sequence of numbers such that $\sum_{\mathbf{m} \in Z_+^n} |a_{\mathbf{m}}|^2 < \infty$, and $z^{\mathbf{m}}$ means the product $\prod_{i=1}^n z_i^{m_i}$ [2]. For a function $f \in L^\infty(D^n)$, f is pluriharmonic. Then we can write the f as

$$f = f_1 + \bar{f}_2, \qquad \text{where } f_1 \text{ and } f_2 \text{ are both in } H^2(D^n)\ [5].$$

2. THE SEMI-COMMUTATOR OF TOEPLITZ OPERATORS

THEOREM 2.1. *Suppose that f and g are bounded pluriharmonic functions on the polydisk D^n for $n > 1$. For any $\mu \in T^{n-1}$, $z' \in D^{n-1}$, and any $z_1 \in D$, we have*

$$\lim_{z' \to \mu'} \int_S \langle T_f T_g k_{z_1} k_{z' e^{i\theta}}, k_{z_1} k_{z' e^{i\theta}} \rangle e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta$$

$$= \int_S \langle T_{f(\cdot \mu' e^{i\theta})} T_{g(\cdot \mu' e^{i\theta})} k_{z_1}, k_{z_1} \rangle e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta,$$

where $T_{f(\cdot, \mu' e^{i\theta})}$ and $T_{g(\cdot \mu' e^{i\theta})}$ are Toeplitz operators on $L_a^2(D)$,

$$\boldsymbol{\theta} = (\theta_2, \dots, \theta_n) \in [0, 2\pi]^{n-1} = S, \qquad \mathbf{m} = (m_2, \dots, m_n) \in Z^{n-1},$$

$$e^{i\mathbf{m}\boldsymbol{\theta}} = e^{i m_2 \theta_2} \cdots e^{i m_n \theta_n} \qquad z' e^{i\theta} = \left(z_2 e^{i\theta_2}, \dots, z_n e^{i\theta_n}\right).$$

Proof. We write $f = f_1 + f_2$, $g = g_1 + g_2$; f_j and g_j are in $H^2(D^n)$. Then

$$\langle T_f T_g k_z, k_z \rangle = \langle T_g k_z, T_{\bar{f}} k_z \rangle = \langle (g_1 + \bar{g}_2(z)) k_z, \left(\bar{f}_1 + f_2\right) k_z \rangle$$

$$= g_1(z) f_1(z) + \bar{g}_2(z) f_1(z) + \bar{g}_2(z) \bar{f}_2(z)$$

$$+ \int_{D^n} g_1(w) \bar{f}_2(w) |k_z(w)|^2\, dA(w).$$

Replacing z by $(z_1, z' e^{i\theta})$ in the above equation and multiplying by $e^{i\mathbf{m}\boldsymbol{\theta}}$ and then integrating with respect to θ yields

$$\int_S \langle T_f T_g k_{z_1} k_{z' e^{i\theta}}, k_{z_1} k_{z' e^{i\theta}} \rangle e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta$$

$$= \int_S g_1\left(z_1, z' e^{i\theta}\right) f_1\left(z_1, z' e^{i\theta}\right) e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta$$

$$+ \int_S \bar{g}_2\left(z_1, z' e^{i\theta}\right) f_1\left(z_1, z' e^{i\theta}\right) e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta$$

$$+ \int_S \bar{g}_2\left(z_1, z' e^{i\theta}\right) \bar{f}_2\left(z_1, z' e^{i\theta}\right) e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta$$

$$+ \int_{D^n} \left(\int_S g_1\left(w_1, w' e^{i\theta}\right) \bar{f}_2\left(w_1, w' e^{i\theta}\right) e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta \right) |k_z(w)|^2\, dA(w).$$

We write the power series of g_j and f_j as

$$g_j(z) = \sum_{\alpha} \hat{g}_j(\alpha, z_1) z'^{\alpha}, \qquad f_j(z) = \sum_{\alpha} \hat{f}_j(\alpha, z_1) z'^{\alpha},\ j = 1, 2.$$

Thus we have

$$\begin{aligned} H_m(z) &= \int_S g_1\left(z_1, z' e^{i\theta}\right) \bar{f}_2\left(z_1, z' e^{i\theta}\right) e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta \\ &= \sum_{\alpha - \beta + \mathbf{m} = 0} \hat{g}_1(\alpha, z_1) \hat{f}_2(\beta, z_1) z'^{\alpha} \bar{z}'^{\beta}, \\ |H_m(z)| &\leq \sum_{\alpha - \beta + \mathbf{m} = \bar{0}} |\hat{g}_1(\alpha, z_1) \hat{f}_2(\beta, z_1) z'^{\alpha} z'^{\beta}| \\ &\leq \sum_{\alpha - \beta + \mathbf{m} = \bar{0}} |\hat{g}_1(\alpha, z_1) \hat{f}_2(\beta, z_1)| \\ &\leq \left(\sum_{\alpha} |\hat{g}_1(\alpha, z_1)|^2 \right)^{1/2} \left(\sum_{\alpha} |\hat{f}_2(\alpha, z_1)|^2 \right)^{1/2}. \end{aligned}$$

Because g_1 and f_2 are all in the Hardy space $H^2(D^n)$, and

$$\begin{aligned} \|g_1\|_{H^2}^2 &= \int_T \int_{T^{n-1}} \left| \sum_{\alpha} \hat{g}_1(\alpha, z_1) z'^{\alpha} \right|^2 d\sigma(z')\, d\sigma(z_1) \\ &= \int_T \sum_{\alpha} |\hat{g}_1(\alpha, z_1)|^2\, d\sigma(z_1) \end{aligned}$$

and

$$\|f_2\|_{H^2}^2 = \int_T \sum_{\alpha} |\hat{f}_2(\alpha, z_1)|^2\, d\sigma(z_1),$$

we thus have

$$\left(\sum_{\alpha} |\hat{g}_1(\alpha, z_1)|^2 \right)^{1/2} \left(\sum_{\alpha} |\hat{f}_2(\alpha, z_1)|^2 \right)^{1/2} \in L^2(T).$$

Thus $H_m(z)$ is continuous on $\overline{D^{n-1}}$ in variables z' for fixed z_1. It is also easy to see that

$$\|\hat{g}_1(\alpha, z_1)\|_{L^2_\alpha}^2 = \int_D |\hat{g}_1(\alpha, z_1)|^2 \, dA(z_1) \le \|\hat{g}_1(\alpha, z_1)\|_{H^2}^2$$
$$= \int_T |\hat{g}_1(\alpha, z_1)|^2 \, d\alpha(z_1).$$

The consideration for $\hat{f}_2(\alpha, z_1)$ is similar to $\hat{g}_1(\alpha, z_1)$.

Hence

$$\left(\sum_\alpha |\hat{g}_1(\alpha, z_1)|^2\right)^{1/2} \left(\sum_\alpha |\hat{f}_2(\alpha, z_1)|^2\right)^{1/2} \in L^2(D) \subset L^2(D^n).$$

For any sequence $\{z'_\alpha = (z_2, \dots, z_n)_\alpha\} \subset D^{n-1}$ converging to $\mu' = \{\mu_2, \dots, \mu_n\}$, by the dominated convergence theorem, we have

$$\lim_{z'_\alpha \to \mu'} \int_{D^n} \int_S g_1(w_1, w' e^{i\theta}) \bar{f}_2(w_1, w' e^{i\theta}) e^{i\mathbf{m}\boldsymbol{\theta}} \, d\theta |k_2(w)|^2 \, dA(w)$$
$$= \lim_{z'_\alpha \to \mu'} \int_{D^n} H(w_1, w') |k_z(w)|^2 \, dA(w)$$
$$= \lim_{z'_\alpha \to \mu'} \int_{D^n} H\big(w_1, \phi_{(z_2)_\alpha}, \dots, \phi_{(z_n)_\alpha}\big) |k_{z_1}(w_1)|^2 \, dA(w)$$
$$\big(\text{because } \phi_{(z_1)_\alpha} \to \mu_j \text{ as } (z_j)_\alpha \to \mu_j \in \partial D\big)$$
$$= \int_{D^n} H(w_1, \mu_2, \dots, \mu_n) |k_{z_1}(w_1)|^2 \, dA(w)$$
$$= \int_D \int_S g_1(w_1 \, \mu' e^{i\theta}) \bar{f}_2(w_1, \mu' e^{i\theta}) e^{i\mathbf{m}\boldsymbol{\theta}} \, d\theta |k_{z_1}(w_1)|^2 \, dA(w_1)$$
$$= \int_S \int_D g_1(w_1, \mu' e^{i\theta}) \bar{f}_2(w_1, \mu' e^{i\theta}) |k_{z_1}(w_1)|^2 \, dA(w_1) e^{i\mathbf{m}\boldsymbol{\theta}} \, d\theta.$$

This implies that

$$\lim_{z' \to \mu'} \int_{D^n} \int_S g_1(w_1, w' e^{i\theta}) \bar{f}_2(w_1, w' e^{i\theta}) e^{i\mathbf{m}\boldsymbol{\theta}} \, d\theta |k_2(w)|^2 \, dA(w)$$
$$= \int_S \int_D g_1(w_1, w' e^{i\theta}) \bar{f}_2(w_1, w' e^{i\theta}) |k_{z_1}(w_1)|^2 \, dA(w_1) e^{i\mathbf{m}\boldsymbol{\theta}} \, d\theta.$$

Similarly we can show that $\int_S g_1(z_1, z'e^{i\theta})f_1(z_1, z'e^{i\theta})e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta$,

$$\int_S \bar{g}_2\big(z_1, z'e^{i\theta}\big)f_1\big(z_1, z'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \qquad \text{and}$$

$$\int_S \bar{g}_2\big(z_1, z'e^{i\theta}\big)\bar{f}_2\big(z_1, z'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta$$

are continuous on the closure $\overline{D^{n-1}}$ in variables z' for fixed z_1. Hence we obtain that

$$\begin{aligned}
&\lim_{z'\to\mu'}\int_S \langle T_f T_g k_{z_1}k_{z'e^{i\theta}}, k_{z_1}k_{z'e^{i\theta}}\rangle e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\quad= \lim_{z'\to\mu'}\int_S g_1\big(z_1, z'e^{i\theta}\big)f_1\big(z_1, z'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\qquad+ \lim_{z'\to\mu'}\int_S g_2\big(z_1, z'e^{i\theta}\big)f_1\big(z_1, z'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\qquad+ \lim_{z'\to\mu'}\int_S \bar{g}_2\big(z_1, z'e^{i\theta}\big)\bar{f}_2\big(z_1, z'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\qquad+ \lim_{z'\to\mu'}\int_{D^n}\int_S g_1\big(w_1, w'e^{i\theta}\big)\bar{f}_2\big(w_1, w'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta|k_z(w)|^2\,dA(w) \\
&\quad= \int_S g_1\big(z_1, \mu'e^{i\theta}\big)f_1\big(z_1, \mu'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\qquad+ \int_S \bar{g}_2\big(z_1, \mu'e^{i\theta}\big)f_1\big(z_1, \mu'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\qquad+ \int_S \bar{g}_2\big(z_1, \mu'e^{i\theta}\big)\bar{f}_2\big(z_1, \mu'e^{i\theta}\big)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\qquad+ \int_S\int_D g_1\big(w_1, \mu'e^{i\theta}\big)\bar{f}_2\big(w_1, \mu'e^{i\theta}\big)|k_{z_1}(w_1)|^2\,dA(w_1)e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta \\
&\quad= \int_S \langle T_{f(\cdot,\,\mu'e^{i\theta})}T_{g(\cdot,\,\mu'e^{i\theta})}k_{z_1}, k_{z_1}\rangle e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta.
\end{aligned}$$

This completes the proof of the theorem.

COROLLARY 2.2. *Let f and g be bounded pluriharmonic functions on the polydisk D^n, $n > 1$. If $T_f T_g$ is compact, then $f(\mu)g(\mu) = 0$ for almost all $\mu \in T^n$.*

Proof. Since k_z weakly converges to zero in $L_a^2(D^n)$ as z tends to the boundary of D^n, if $T_f T_g$ is compact, then $\lim_{z' \to \mu'} T_f T_g k_z = 0$ for any $z' \in D^{n-1}$ and $\mu' \in T^{n-1}$. It follows from Theorem 2.1 that $\int_S \langle T_{f(\cdot, \mu' e^{i\theta})} T_{g(\cdot, \mu' e^{i\theta})} k_{z_1}, k_{z_1} \rangle e^{i\mathbf{m}\boldsymbol{\theta}}\, d\theta = 0$ for any $\mathbf{m} \in Z^{n-1}$. The injection of the Fourier transformation implies that $\langle T_{f(\cdot, \mu' e^{i\theta})} T_{g(\cdot, \mu' e^{i\theta})} k_{z_1}, k_{z_1} \rangle = 0$ for any $\mu' \in T^{n-1}$ and almost all $\theta \in S$. That is, $(g_1 f_1)(z_1, \mu') + (\bar{g}_2 f_1)(z_1, \mu') + (\bar{g}_2 f_2)(z_1, \mu') + \langle g_1(\cdot, \mu') f_2(\cdot, \mu') k_{z_1}, k_{z_1} \rangle = 0$ for any $z_1 \in D$ and almost all $\mu' \in T^{n-1}$. Replacing z_1 by $z_1 e^{i\theta_1}$ in the above equation and multiplying by $e^{im\theta_1}$ and then integrating with respect to θ_1 yields

$$\int_0^{2\pi} \left[g_1 f_1 + \bar{g}_2 f_1 + \bar{g}_2 \bar{f}_2 \right] \left(z_1 e^{i\theta_1}, \mu' \right) e^{im\theta_1}\, d\theta_1 + \int_0^{2\pi} \int_D \left(w_1 e^{i\theta_1}, \mu' \right) f_2 \left(w_1 e^{i\theta_1}, \mu' \right) |k_{z_1}(w_1)|^2\, dA(w_1) e^{im\theta_1}\, d\theta_1 = 0.$$

Similarly to Theorem 2.1 we can also obtain

$$\begin{aligned} &\lim_{z_1 \to \mu_1} \int_0^{2\pi} \left[g_1 f_1 + \bar{g}_2 f_1 + \bar{g}_2 \bar{f}_2 \right] \left(z_1 e^{i\theta_1}, \mu' \right) e^{im\theta_1}\, d\theta_1 \\ &\quad + \lim_{z_1 \to \mu_1} \int_0^{2\pi} \int_D g_1 \left(w_1 e^{i\theta_1}, \mu' \right) \bar{f}_2 \left(w_1 e^{i\theta_1}, \mu' \right) |k_{z_1}(w_1)|^2\, dA(w_1) e^{im\theta_1}\, d\theta_1 \\ &\quad = \int_0^{2\pi} \left[g_1 f_1 + \bar{g}_2 f_1 + \bar{g}_2 \bar{f}_2 \right] \left(\mu_1 e^{i\theta_1}, \mu' \right) e^{im\theta_1}\, d\theta_1 \\ &\qquad + \int_0^{2\pi} g_1 \left(\mu_1 e^{i\theta_1}, \mu' \right) f_2 \left(\mu_1 e^{i\theta_1}, \mu' \right) e^{im\theta_1}\, d\theta_1 = 0. \end{aligned}$$

That is, $\int_0^{2\pi} g(\mu_1 e^{i\theta_1}, \mu') f(\mu_1 e^{i\theta_1}, \mu') e^{im\theta_1}\, d\theta_1 = 0$. This implies $f(\mu) g(\mu) = 0$ for almost all $\mu \in T^n$. This completes the proof of the corollary.

THEOREM 2.3. *Let f and g be two bounded pluriharmonic functions on the polydisk D^n for $n > 1$. The following are equivalent*:

(1) $T_f T_g - T_{gf}$ *is zero.*

(2) $T_f T_g - T_{gf}$ *is compact.*

(3) $H_{\bar{f}}^* H_g$ *is compact.*

(4) $\|\langle (T_f T_g - T_{fg}) k_z, k_z \rangle\| \to 0$, *as* $z \to \partial D^n$.

(5) $\langle (T_f T_g - T_{fg}) k_z, k_z \rangle \to 0$, *as* $z \to \partial D^n$.

(6) *For each j ($j = 1, 2, \ldots, n$), if g is not a constant in z_j, then either $\bar{f}$ or g is analytic in z_j.*

Proof. As in the case of the unit disk, the semi-commutator is connected to the Hankel operators by the relation $T_f T_g - T_{fg} = -H_{\bar{f}}^* H_g$. It follows that (2) and (3) are equivalent. Thus we easily see that (1) implies (2), (2) and (3) are equivalent, (3) implies (4), and (4) implies (5).

Now we prove that (6) implies (1). Assume that for each j ($j = 1, 2, \ldots, n$), either $\bar{f}$ or g is analytic in z_j. Without loss of generality, assume that $\bar{f}$ is analytic in z_1 and g is analytic in z_2. Then it is easy to see that

$$(I - P)(gh_1) = (I_1 - P_1)(gh_1) \text{ for all } h_1 \in L_a^2(D'')$$
$$\text{and} \qquad (1 - P)\left(\bar{f}h_2\right) = (I_2 - P_2)\left(\bar{f}h_2\right)$$

for all $h_2 \in L_a^2(D^n)$, where I_1 is the identity on $L^2(D_1 \times D_3 \times \cdots \times D_n)$ and P_1 is the orthogonal projection from $L^2(D_1 \times D_3 \times \cdots \times D_n)$ onto $L_a^2(D_1 \times D_3 \times \cdots \times D_n)$. I_2 is the identity on $L^2(D_2 \times D_3 \times \cdots \times D_n)$ and P_2 is the orthogonal projection from $L^2(D_2 \times D_3 \times \cdots \times D_n)$ onto $L_a^2(D_2 \times D_3 \times \cdots \times D_n)$ and $D_j = D$, $j = 1, 2, \ldots, n$. Therefore

$$\langle H_g h_1, H_{\bar{f}} \rangle = 0 \qquad \text{for all } h_1, h_2 \in L_a^2(D'').$$

That is, $H_{\bar{f}}^* H_g = 0$. So $T_f T_g - T_{fG} = -H_{\bar{f}}^* H_g = 0$.

Next we prove that (5) implies (6). Without loss of generality, we only have to prove that either $\bar{f}$ or g is analytic in z_1. We write $f = f_1 + \bar{f}_2$, $g = g_1 + \bar{g}_2$, and f_j and g_j all in $H^2(D^n)$. Then

$$\langle (T_f T_g - T_{fg}) k_z, k_z \rangle = \bar{g}_2(z) f_1(z) - \int_{D^n} f_1(w) \bar{g}_2(w) |k_z(w)|^2 \, dA(w).$$

Let $\boldsymbol{\theta} = (\theta_2, \ldots, \theta_n) \in [0, 2\pi]^{n-1} = S$, $\mathbf{m} = (m_2, \ldots, m_n) \in Z^{n-1}$,

$$e^{i\mathbf{m}\boldsymbol{\theta}} = e^{im_2\theta_2} \cdots e^{im_n\theta_{n'}}, \qquad z' e^{i\theta} = \left(z_2 e^{i\theta_2}, \ldots, z_n e^{i\theta_n}\right).$$

Replacing z by $(z_1, z' e^{i\theta})$ in the above equation and multiplying by $e^{i\mathbf{m}\boldsymbol{\theta}}$ and then integrating with respect to θ yields

$$\int_S \langle (T_f T_g - T_{fg}) k_{z_1(z' e^{i\theta})}, k_{z_1(z' e^{i\theta\theta})} \rangle e^{i\mathbf{m}\boldsymbol{\theta}} \, d\boldsymbol{\theta}$$
$$= \int_S \bar{g}_2\left(z_1, z' e^{i\theta}\right) f_1\left(z_1, z' e^{i\theta}\right) e^{i\mathbf{m}\boldsymbol{\theta}} \, d\theta$$
$$- \int_{D^n} \int_S \bar{g}_2\left(w_1, w' e^{i\boldsymbol{\theta}}\right) f_1\left(w_1, w' e^{i\boldsymbol{\theta}}\right) e^{i\mathbf{m}\boldsymbol{\theta}} \, d\boldsymbol{\theta} |k_z(w)|^2 \, dA(w).$$

Using the same method as in the proof of Theorem 2.1, we have

$$\begin{aligned}\lim_{z'\to\mu'}&\int_S\langle(T_fT_g-T_{fg})k_{z_1(z'e^{i\theta})},k_{z_1(z'e^{i\theta})}\rangle e^{i\mathbf{m\theta}}\,d\theta\\ &=\int_S\bar{g}_2(z_1,\mu'e^{i\theta})f_1(z_1,\mu'e^{i\theta})e^{i\mathbf{m\theta}}\,d\theta\\ &\quad-\int_S\int_D\bar{g}_2(w_1,\mu'e^{i\theta})f_1(w_1,\mu'e^{i\theta})|k_{z_1}(w_1)|^2\,dA(w_1)e^{i\mathbf{m\theta}}\,d\theta.\end{aligned}$$

Because $\lim_{z'\to\mu'}\langle(T_fT_g-T_{fg})k_z,k_z\rangle=0$, then

$$\lim_{z'\to\mu'}\int_S\langle(T_fT_g-T_{fg})k_{z_1(z'e^{i\theta})},k_{z_1(z'e^{i\theta})}\rangle e^{i\mathbf{m\theta}}\,d\theta=0.$$

It follows that

$$\int_S\Big[\bar{g}_2(z_1,\mu'e^{i\theta})f_1(z_1,\mu'e^{i\theta})-\int_D\bar{g}_2(w_1,\mu'e^{i\theta})f_1(w_1,\mu'e^{i\theta})|k_{z_1}(w_1)|^2\,dA(w_1)\Big]e^{i\mathbf{m\theta}}\,d\theta=0.$$

The injection of the Fourier transformation implies that $\int_D f_1(w_1,\mu')g_2(w_1,\mu')|k_{z_1}(w_1)|^2\,dA(w_1)=f_1(z_1,\mu')g_2(z_1,\mu')$ for almost all $\mu'\in T^{n-1}$. That is, the function $F(z_1)=f_1(z_1,\mu')g_2(z_1,\mu')$ has the area version of the invariant mean value property. Using the same method as in the proof of Theorem 1 in [1], it is easy to prove $F(z_1)$ is harmonic on D. This implies that $\partial f_1(z_1,\mu')/\partial z_1=0$ or $\partial g_2(z_1,\mu')/\partial z_1=0$ for each $z_1\in D$ and for almost all $\mu'\in T^{n-1}$. It follows that either f or g is analytic in z_1. This finishes the proof of the theorem.

COROLLARY 2.4. *Assume that g is a bounded pluriharmonic function on the polydisk D^n ($n>1$). Then the Hankel operator H_g is compact if and only if g is a constant on D^n.*

Note that when $n=1$, the Hankel operator H_g on $L_a^2(D)$ is compact for any $g\in C(\overline{D})$ (see [3]). So Corollary 2.4 is false on the disk.

3. THE COMMUTATOR OF TOEPLITZ OPERATORS

THEOREM 3.1. *Suppose that f and g are bounded pluriharmonic functions on the polydisk D^n, $n>1$. We write f and g as $f=f_1+f_2$ and $g=g_1+g_2$, f_j and g_j both in $H^2(D^n)$. If the T_f with T_g are essentially commuting, then* *the following holds. For all $z_j \in D$ $(1 \leq j \leq n)$ and almost all $z_i \in T$ when $i \neq j$,*

$$\frac{\partial f_1}{\partial z_j}(z_1,\ldots,z_n)\overline{\frac{\partial g_2}{\partial z_j}}(z_1,\ldots,z_n) = \frac{\partial g_1}{\partial z_j}(z_1,\ldots,z_n)\overline{\frac{\partial f_2}{\partial z_j}}(z_1,\ldots,z_n). \tag{3.1}$$

Proof. If the T_f with T_g are essentially commuting, then

$$0 = \lim_{z'\to\mu'}\int_S \langle (T_fT_g - T_gT_f)k_{z_1(z'e^{i\theta})}, k_{z_1(z'e^{i\theta})}\rangle e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta$$

$$= \int_S \langle (T_{f(\cdot,\,\mu'e^{i\theta})}T_{g(\cdot,\,\mu'e^{i\theta})} - T_{g(\cdot,\,\mu'e^{i\theta})}T_{f(\cdot,\,\mu'e^{i\theta})})k_{z_1}, k_{z_1}\rangle e^{i\mathbf{m}\boldsymbol{\theta}}\,d\theta$$

by Theorem 2.1. This implies that $\langle (T_{f(\cdot,\,\mu')}T_{g(\cdot,\,\mu')} - T_{g(\cdot,\,\mu')}T_{f(\cdot,\,\mu')}k_{z_1}, k_{z_1}\rangle = 0$ for almost all $\mu' \in T^{n-1}$. That is,

$$\left\langle (g_1\bar{f}_2 - f_1\bar{g}_2)k_{z_1}, k_{z_1}\right\rangle + (f_1\bar{g}_2 - g_1\bar{f}_2)(z_1,\mu') = 0, \qquad \text{i.e.}$$

$$\int_D (g_1\bar{f}_2 - f_1\bar{g}_2)(w_1,\mu')|k_{z_1}(w_1)|^2\,dA(w_1) = (g_1\bar{f}_2 - f_1\bar{g}_2)(z_1,\mu').$$

That is, also $\int_D (g_1\bar{f}_2 - f_1\bar{g}_2)(\Phi_{z_1}(w_1),\mu')\,dA(w_1) = (g_1\bar{f}_2 - f_1\bar{g}_2)(\Phi_{z_1}(0),\mu')$, where $\Phi_{z_1} \in \mathrm{Aut}(D)$ and $\forall z_1 \in D$. Thus the function $F(z_1) = (g_1\bar{f}_2 - f_1\bar{g}_2)(z_1,\mu')$ has the area version of the invariant mean value property. Using the argument from Axler and Čučković [1], we can prove that $F(z_1)$ is a harmonic in z_1. It follows that

$$\frac{\partial f_1}{\partial z_1}(z_1,\mu')\overline{\frac{\partial g_2}{\partial z_1}}(z_1,\mu') = \frac{\partial g_1}{\partial z_1}(z_1,\mu')\overline{\frac{\partial f_2}{\partial z_1}}(z_1,\mu')$$

for every $z_1 \in D$ and $\mu' \in T^{n-1}$. This completes the proof of the theorem.

THEOREM 3.2. *Let f be a bounded pluriharmonic function on D^n for $n > 1$. For every j $(1 \leq j \leq n)$, either g or $\bar{g}$ is analytic in z_j. The following are equivalent*:

(1) $T_fT_g = T_gT_f$.

(2) $T_fT_g - T_gT_f$ *is compact.*

(3) *For each j $(j = 1, 2, \ldots, n)$, if g is not a constant in z_j, then either f and g are analytic in z_j or $\bar{f}$ and $\bar{g}$ are analytic in z_j.*

Proof. It is obvious that (1) implies (2). Since $T_f T_g - T_g T_f = H_{\bar{g}}^* H_{\bar{f}} - H_{\bar{f}}^* H_{\bar{g}}$, if (3) holds, then using the argument as in Theorem 2.2, we can show that $H_{\bar{g}}^* H_f = 0$ and $H_{\bar{f}}^* H_g = 0$. That is, $T_f T_g = T_g T_f$. Thus (3) implies (1).

Now we show that (2) implies (3). Assume that $T_f T_g - T_g T_f$ is compact. For the sake of convenience, assume that g is analytic in z_1. We prove that f is also analytic in z_1. Because $T_f T_g - T_g T_f$ is compact, then similar to the proof of Theorem 2.2, we can show that $\bar{f}_2(z_1, \mu')g_1(z_1, \mu')$ is harmonic in z_1 on D for almost all $\mu' \in T^{n-1}$. Hence $(\partial g_1/\partial z_1)(z_1, \mu')(\overline{\partial f_2}/\partial z_1)(z_1, \mu') = 0$, where $z_1 \in D$ and $\mu' \in T^{n-1}$. So if g is not a constant in z_1, then $(\overline{\partial f_2}/\partial z_1)(z_1, \mu') = 0$. That is, f is analytic in z_1. This proves Theorem 3.2.

THEOREM 3.3. *Suppose that f and g are bounded pluriharmonic functions on D^n for $n > 1$ and for every j $(1 \le j \le n)$, $f(z^{(j)}) = 0$ and $g(z^{(j)}) = 0$, where $z^{(i)} = (z_1, \dots, z_{j-1}, 0, z_{j+1} \dots z_n) \in D^n$. The following are equivalent*:

(1) $T_f T_g = T_g T_f$.

(2) $T_f T_g - T_g T_f$ *is compact.*

(3) *For every j $(1 \le j \le n)$, either f and g are analytic in z_j or $\bar{f}$ and $\bar{g}$ are analytic in z_j or there exist constants C such that $f = Cg$.*

Proof. We only prove that (2) implies (3). Assume $T_f T_g - T_g T_f$ is compact. Without loss of generality, let $j = 1$ to prove that (3) holds. By Theorem 3.1, we have

$$\frac{\partial f_1}{\partial z_1}(z_1, \mu')\,\overline{\frac{\partial g_2}{\partial z_1}}(z_1, \mu') = \frac{\partial g_1}{\partial z_1}(z_1, \mu')\,\overline{\frac{\partial f_2}{\partial z_1}}(z_1, \mu').$$

If $(\partial g_2/\partial z_1)(z_1, \mu') = 0$, then either $\partial g_1/\partial z_1 = 0$ or $\partial f_2/\partial z_1 = 0$. This implies that either g is a constant in z_1 or f and g are analytic in z_1. Hence the condition (3) holds. Now assume that $\partial g_2/\partial z_1 \ne 0$ and $\partial g_1/\partial z_1 \ne 0$. Then we have

$$\frac{(\partial f_1/\partial z_1)(z_1, \mu')}{(\partial g_1/\partial z_1)(z_1, \mu')} = \overline{\left(\frac{(\partial f_2/\partial z_1)(z_1, \mu')}{(\partial g_2/\partial z_1)(z_1, \mu')}\right)} = C(\mu') \qquad (3.2)$$

for all $z_1 \in D$ and $\mu' = (\mu_2, \dots, \mu_n) \in T^{n-1}$.

By the assumption of the theorem, we can write

$$f_j = \sum_{m=1}^{\infty} \hat{f}_j(m, z')z_1^m \quad \text{and} \quad g_j = \sum_{m=1}^{\infty} \hat{g}_j(m, z')z_1^m, \qquad j = 1,2.$$

Equation (3.2) is

$$\frac{\sum_{m=1}^{\infty} m\hat{f}_1(m,\mu')z_1^{m-1}}{\sum_{m=1}^{\infty} m\hat{g}_1(m,\mu')z_1^{m-1}} = \frac{\sum_{m=1}^{\infty} m\overline{\hat{f}_2(m,\mu')z_1^{m-1}}}{\sum_{m=1}^{\infty} m\overline{\hat{g}_2(m,\mu')z_1^{m-1}}} = C(\mu').$$

This implies that

$$\sum_{m=1}^{\infty} m\hat{f}_1(m,\mu')z_1^{m-1} = C(\mu')\sum_{m=1}^{\infty} m\hat{g}_1(m,\mu')z_1^{m-1},$$
$$\sum_{m=1}^{\infty} m\hat{f}_2(m,\mu')z_1^{m-1} = \overline{C(\mu')}\sum_{m=1}^{\infty} m\hat{g}_2(m,\mu')z_1^{m-1}.$$

Thus we have

$$\hat{f}_1(m,\mu') = C(\mu')\hat{g}_1(m,\mu') \qquad \text{and} \qquad \hat{f}_2(m,\mu') = \overline{C(\mu')}\hat{g}_2(m,\mu').$$

Hence

$$\hat{f}_1(m,\mu')z_1^m = C(\mu')\hat{g}_1(m,\mu')z_1^m \qquad \text{and}$$
$$\hat{f}_2(m,\mu')z_1^m = \overline{C}(\mu')\hat{g}_2(m,\mu')z_1^m.$$

It follows that

$$\begin{aligned} f_1(z_1,\mu') &= \sum_{m=1}^{\infty}\hat{f}_1(m,\mu')z_1^m = C(\mu')\sum_{m=1}^{\infty}\hat{g}_1(m,\mu')z_1^m \\ &= C(\mu')g_1(z_1,\mu'), \\ f_2(z_1,\mu') &= \sum_{m=1}^{\infty}\hat{f}_2(m,\mu')z_1^m = C(\mu')\sum_{m=1}^{\infty}\hat{g}_2(m,\mu')z_1^m \\ &= \overline{C}(\mu')g_2(z_1\,\mu'). \end{aligned}$$

This implies that

$$\frac{f_1(\mu,\mu')}{g_1(\mu_1,\mu')} = \frac{\bar{f}_2(\mu_1,\mu')}{\bar{g}_2(\mu_1,\mu')} = C(\mu') \tag{3.3}$$

is a constant in μ_1. If f and g are analytic in z_2 or $\bar{f}$ and $\bar{g}$ are analytic in z_2, then clearly $C(\mu') = C(\mu'')$ is a constant in μ_2 by Eq. (3.3). If neither f and g are analytic in z_2 nor $\bar{f}$ and $\bar{g}$ are analytic in z_2, using the same argument as above we can show

$$\frac{(\partial f_1/\partial z_2)(z_2, \mu'')}{(\partial g_1/\partial z_2)(z_2, \mu'')} = \frac{(\overline{\partial f_2}/\partial z_2)(z_2, \mu'')}{(\overline{\partial g_2}/\partial z_2)(z_2, \mu'')} = D(\mu'') \tag{3.4}$$

is a constant in μ_2, where $\mu'' = (\mu_1, \mu_3, \ldots, \mu_n) \in T^{n-1}$. This implies that

$$\sum_{m=1}^{\infty} m\hat{f}_1(m, \mu'') z_2^{m-1} = D(\mu'') \sum_{m=1}^{\infty} m\hat{g}_1(m, \mu'') z_2^{m-1},$$

$$\sum_{m=1}^{\infty} m\hat{f}_2(m, \mu'') z_2^{m-1} = \overline{D(\mu')} \sum_{m=1}^{\infty} m\hat{g}_2(m, \mu'') z_2^{m-1}.$$

It follows that $\hat{f}_1(m, \mu'') = D(\mu'')\hat{g}_1(m, \mu'')$, $\hat{f}_2(m, \mu'') = \overline{D}(\mu'')$ $\hat{g}_2(m, \mu'')$. Hence

$$\begin{aligned} f_1(\mu_1, z_2, \ldots, \mu_n) &= \sum_{m=1}^{\infty} \hat{f}_1(m, \mu'') z_2^m \\ &= D(\mu'') \sum_{m=1}^{\infty} \hat{g}_1(m, \mu'') z_2^m \\ &= D(\mu'') g_1(\mu_1, z_2, \ldots, \mu_n), \\ f_2(\mu_1, z_2, \ldots, \mu_n) &= \overline{D}(\mu'') g_2(\mu_1, z_2, \ldots, \mu_n). \end{aligned}$$

Thus we have

$$\frac{f_1(\mu_1, \mu_2, \ldots, \mu_n)}{g_1(\mu_1, \mu_2, \ldots, \mu_n)} = \frac{\overline{f_2(\mu_1, \mu_2, \ldots, \mu_n)}}{\overline{g_2(\mu_1, \mu_2, \ldots, \mu_n)}} = D(\mu'')$$

is a constant in μ_2. Going on, we can obtain that

$$\frac{f_1(\mu_1, \mu_2, \ldots, \mu_n)}{g_1(\mu_1, \mu_2, \ldots, \mu_n)} = \frac{\bar{f}_2(\mu_1, \mu_2, \ldots, \mu_n)}{\bar{g}_2(\mu_1, \mu_2, \ldots, \mu_n)}$$

is a constant in all invariables z_j $(1 \le j \le n)$. Put

$$\frac{f_1(\mu_1, \mu_2, \ldots, \mu_n)}{g_1(\mu_1, \mu_2, \ldots, \mu_n)} = \frac{\bar{f}_2(\mu_1, \mu_2, \ldots, \mu_n)}{\bar{g}_2(\mu_1, \mu_2, \ldots, \mu_n)} = C.$$

Then we have $f_1(z) = Cg_1(z)$ and $f_2(z) = \overline{C}g_2(z)$ for every $z \in D^n$. It follows that $f(z) = Cg(z)$. This completes the proof of the theorem.

Remark. It is natural to guess that when replacing either g or $\bar{g}$ is analytic by g is pluriharmonic, Theorem 3.3 is still true. But we can't prove that now.

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